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Chapter 1

Many classes of optimal Jarratt-type fourth-order methods

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Abstract: In this paper, we have presented a family of two-point fourth order iterative methods without memory based on power mean using a weight function. The family of fourth order methods is optimal in the sense of Kung-Traub hypothesis. In terms of computational point of view, our methods require three evaluations (one function and two first derivatives) to get fourth order. Hence, these methods have high efficiency index 1.5874. Some known results can be regarded as particular cases of our family of methods. Some numerical examples are tested to know the performance of the methods which verifies the theoretical results.

Keywords: Non-linear equation, Iterative Methods, Newton's Method, order of convergence, power mean.

1. Introduction

Consider the problem of finding a simple root of a single non-linear equation of the form:

$$f(x) = 0, \tag{1}$$

where $f : I \subseteq R \to R$ for an open interval I is a scalar function. Solution of equation (1) has been given much attention because of its importance in many branches of engineering and science. One of familiar method to solve (1) is the following second order Newton's method:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ where n = 0, 1, 2…} \tag{2}$$

In recent years, cubic convergence was established for solving (1) by many methods. In [11] Xiaojian, derived a class of Newton's methods based on power means by using the trapezoidal formula

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ z_n = x_n - \dfrac{2^{1/p} f(x_n)}{sign\left(f'(x_n)\right)\left(f'(x_n)^p + f'(y_n)^p\right)^{\frac{1}{p}}} \end{cases} . \tag{3}$$

For a finite real number p that is p ∈ R, setting p = 1, − 1, 0, 2, etc. we get several particular cases of well-known means which are respectively called arithmetic mean, harmonic mean, geometry mean and square mean and etc..

In [5] Jarratt, developed fourth order method in which one evaluation of the functions and two evaluation of the first derivative is defined by

$$\begin{cases} y_n = x_n - \dfrac{2f(x_n)}{3f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}\dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{4}$$

In [6] Khattri et al, recently derived an efficient fourth-order technique, in which three evaluation function

$$\begin{cases} y_n = x_n - \dfrac{2f(x_n)}{3f'(x_n)}, \\ x_{n+1} = x_n - \left[1+\dfrac{21}{8}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)-\dfrac{9}{2}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)^2+\dfrac{15}{8}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)^3\right]\dfrac{f(x_n)}{f'(x_n)} \end{cases} \tag{5}$$

Equation (4) and (5) are optimal methods without memory in viewpoint of Kung-Trub conjecture [7].

2. Description of the Jarratt-type fourth order methods

Let us consider the power mean Newton's method (3), which is order three with three evaluations per full iteration. Clearly its efficiency index $E^* = 1.442$ is not high optimal, use of weight function approach in (3) to build our generalized class of optimal fourth method and little change in first step of (3). Thus, we consider

$$\begin{cases} y_n = x_n - \dfrac{2f(x_n)}{3f'(x_n)} \\ x_{n+1} = x_n - \dfrac{2^{1/p}\, f(x_n)}{sign\left(f'(x_n)\right)\left(f'(x_n)^p + f'(y_n)^p\right)^{\frac{1}{p}}}\left[G(t)\times H(\tau)\right], \end{cases} \tag{6}$$

Where $G(t)$ and $H(\tau)$ are two real valued functions and p is a finite real number that is p ∈ R. where $t = \dfrac{f(x)}{f'(x)}$ and $\tau = \dfrac{f'(y)}{f'(x)}$. Different value of p and chosen weight function such that the order convergence is optimal level fourth order without using more evaluations of the function. In [9], Soleymani et al. presented two new classes of optimal Jarratt-type fourth order methods which is particular case of (6) that is p = 1 and p = -1 are arithmetic mean and harmonic mean respectively. In this paper we consider only p = 0 and p = 2 are geometry mean and square mean respectively. Note that for any value of p we obtain a class of optimal Jarratt-type fourth order method with under the choice of condition of the weight functions.

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Theorem 2.1: Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq R \to R$ for an open interval *I*. Then the class of methods without memory (6) is of fourth-order convergence.
Proof.

Let α be a simple zero of the function $f(x)=0$. (That is, $f(\alpha)=0$ and $f'(\alpha)\neq 0$).

By expanding $f(x_n)$ and $f'(x_n)$ by Taylor series about α, we obtain

$$f(x_n) = f'(\alpha)\,[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5)], \tag{7}$$

where $c_k = \dfrac{f^{(k)}(\alpha)}{k!\,f'(\alpha)}$, k = 2, 3, 4 … and $e_n = x_n - \alpha$.

$$f'(x_n) = f'(\alpha)\,[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4)], \tag{8}$$

Using (7), (8) in first step of (6) we get,

$$y_n = \alpha + \frac{e_n}{3} + \frac{2c_2 e_n^2}{3} - \frac{4}{3}(c_2^2 - c_3)e_n^3 + \frac{2}{3}(4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + o(e_n^5) \tag{9}$$

Applying the Taylor series of (9) we have,

$$f'(y_n) = f'(\alpha)\,[1 + \tfrac{2}{3}c_2 e_n + \tfrac{1}{3}(4c_2^2 + c_3)e_n^2 + \tfrac{4}{27}(-18c_2^3 + 27c_2c_3 + c_4)e_n^3 + O(e_n^4)], \tag{10}$$

Using (7), (8) and (10) we get,

$$\frac{f(x_n)}{sign(f'(x_n))\left(\dfrac{f'(x_n)^p + f'(\mathrm{y}_n)^p}{2}\right)^{\frac{1}{p}}} = \left[e_n - \frac{1}{3}c_2 e_n^2 - \left(\frac{2}{3}c_3 + \frac{2}{9}pc_2^2\right)e_n^3 - \left(\frac{29}{27}c_4 + \left(\frac{8}{9}p - \frac{1}{3}\right)c_2c_3 + \left(\frac{52}{81}p^2 - \frac{90}{27}p + \frac{68}{81}\right)c_2^3\right)e_n^4 + O(e_n^5)\right] \tag{11}$$

In (6) expanding the Taylor series of weight functions $G(t)$ and $\mathrm{H}(\tau)$ about 0 and 1 respectively, then multiply in (11) and then with some choice condition we will obtain optimal Jarratt-type fourth order methods.

Case A: When p = 0 in (6), and satisfying this conditions

$$\begin{cases} G(0)=1, \quad G'(0)=0=G''(0), \quad \left|G^{(3)}(0)\right| \le +\infty \\ \mathrm{H}(1)=1, \quad H'(1)=-\dfrac{1}{4}, \quad H''(1)=\dfrac{5}{4}, \quad \left|H^{(3)}(1)\right| \le +\infty \end{cases} \tag{12}$$

Using (12) in (6), we obtain new class of optimal fourth order method, the error equation is

$$e_{n+1} = \left[\frac{1}{81}(395 + 32H^{(3)}(1))c_2^3 - \frac{1}{6}G^{(3)}(0) - c_2c_3 + \frac{1}{9}c_4\right]e_n^4 + o(e_n^5). \tag{13}$$

This means that the general two step methods (6) and (12) reaches fourth order convergence □

Now by choosing appropriate weight functions as presented in (12), we can give optimal two-step fourth-order iterative methods, such as

Method I:

$$\begin{cases} y_n = x_n - \dfrac{2f(x_n)}{3f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{sign(f'(x_n))\sqrt{f'(x_n)f'(y_n)}}\left[\left(1+\left(\dfrac{f(x_n)}{f'(x_n)}\right)^3\right)\left(\dfrac{15}{8}-\dfrac{3}{2}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)+\dfrac{5}{8}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)^2\right)\right] \end{cases} \tag{14}$$

Where its error equation of (14) is

$$e_{n+1} = \left[\frac{395}{81}c_2^3 - 1 - c_2c_3 + \frac{1}{9}c_4\right]e_n^4 + o(e_n^5). \tag{15}$$

Method II:

$$\begin{cases} y_n = x_n - \dfrac{2f(x_n)}{3f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{sign(f'(x_n))\sqrt{f'(x_n)f'(y_n)}}\left[\left(1+\left(\dfrac{f(x_n)}{f'(x_n)}\right)^3\right)\left(\dfrac{7}{8}+\dfrac{3}{2}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)-\dfrac{19}{8}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)^2+\left(\dfrac{f'(y_n)}{f'(x_n)}\right)^3\right)\right] \end{cases} \tag{16}$$

Where its error equation of (16) is

$$e_{n+1} = \left[\frac{587}{81}c_2^3 - 1 - c_2c_3 + \frac{1}{9}c_4\right]e_n^4 + o(e_n^5). \tag{17}$$

Case B: When p = 2 in (6), and satisfying this condition

$$\begin{cases} G(0)=1, \quad G'(0)=0=G''(0), \quad \left|G^{(3)}(0)\right| \le +\infty \\ H(1)=1, \quad H'(1)=-\dfrac{1}{4}, \quad H''(1)=\dfrac{7}{4}, \quad \left|H^{(3)}(1)\right| \le +\infty \end{cases} \tag{18}$$

Using (18) in (6), we get another new class of optimal fourth order method, the error equation is

$$e_{n+1} = \left[\frac{1}{81}(303+32H^{(3)}(1))c_2^3 - \frac{1}{6}G^{(3)}(0) - c_2c_3 + \frac{1}{9}c_4\right]e_n^4 + o(e_n^5). \tag{19}$$

This means that the general two step methods (6) and (18), we obtain fourth order convergence. □

Now by choosing appropriate weight functions as presented in (18), we can give optimal two-step fourth-order iterative methods, such as

Method III:

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$$\begin{cases} y_n = x_n - \dfrac{2f(x_n)}{3f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{sign(f'(x_n))\sqrt{\left(\dfrac{f'(x_n)^2 + f'(y_n)^2}{2}\right)}}\left[\dfrac{17}{8} - 2\left(\dfrac{f'(y_n)}{f'(x_n)}\right) + \dfrac{7}{8}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)^2\right] \end{cases} \tag{20}$$

Where its error equation of (20) is

$$e_{n+1} = \left[\frac{303}{81}c_2^3 - c_2c_3 + \frac{1}{9}c_4\right]e_n^4 + o(e_n^5). \tag{21}$$

Method IV:

$$\begin{cases} y_n = x_n - \dfrac{2f(x_n)}{3f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{sign(f'(x_n))\sqrt{\left(\dfrac{f'(x_n)^2 + f'(y_n)^2}{2}\right)}}\left[\left(1 + \left(\dfrac{f(x_n)}{f'(x_n)}\right)^3\right)\left(\dfrac{17}{8} - 2\left(\dfrac{f'(y_n)}{f'(x_n)}\right) + \dfrac{7}{8}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)^2\right)\right] \end{cases} \tag{22}$$

Where its error equation of (22) is

$$e_{n+1} = \left[\frac{303}{81}c_2^3 - c_2c_3 + \frac{1}{9}c_4 - 1\right]e_n^4 + o(e_n^5). \tag{23}$$

Note: In [9], Soleymani et al. already derived when p = 1 and -1 in (6) with under suitable choice of weight function, which have two class of optimal Jarratt-type fourth-order methods. See [9] the methods are (17), (19), (24) and (25).

Remark 1:

It is clear that our proposed method of iterations requires three function evaluations per iteration, i.e. two first derivative and one function evaluations. Thus, our new method is optimal. Clearly its efficiency index is $E^* = 1.5874$ (high optimal).

3. Numerical Examples

In this section, we give numerical results on some test functions to compare the efficiency of proposed methods, with Newton's method (NM), and some existing sixth order methods given in section 3. Numerical computations have been carried out in the MATLAB software rounding to 16 significant digits. Depending on the precision of the computer, we use the following stopping criteria for the iterative process: $|x_{n+1} - x_n| + |f(x_n)| < \varepsilon,$ where $\varepsilon = 10^{-14}$.

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Table 1: Errors occurring the estimates of the root of function $f_1(x) = xe^{-x} - 0.1$ by the methods described below with initial guess $x_0 = 0.3$, and the root $\alpha \approx$ 0.111832559158963

Methods	$\lvert x_1 - \alpha \rvert$	$\lvert x_2 - \alpha \rvert$	$\lvert x_3 - \alpha \rvert$
Newton Method	0.47567e-01	0.22849e-02	0.55356e-05
Weerakoon, p = 1 in (3)	0.13039e-01	0.31800e-05	0.55511e-16
Homeier, p = -1 in (3)	0.64393e-03	0.72263e-10	0.0
Lukic, p = 0 in (3)	0.69472e-02	0.28266e-06	0.0
Khattri (5)	0.56585e-01	0.78841e-04	0.41633e-15
Soleymani (17) in [9]	0.14507e-01	0.20259e-06	0.13878e-16
Jaiswal (2.25) in [3]	0.11886e-01	0.73037e-07	0.0
Method I - (14)	0.13110e-01	0.12391e-06	0.0
Method II - (16)	0.29305e-01	0.46218e-05	0.0
Method III - (20)	0.13262e-01	0.12540e-06	0.0
Method IV - (22)	0.15900e-01	0.31707e-06	0.0

Note: NM 6 itr, and other 4 itr.

Table 2: Errors occurring the estimates of the root of function $f_2(x) = \dfrac{x^3 + 2.87x^2 - 10.28}{4.62} - x$ by the methods described below with initial guess $x_0 = 2.5$, and the root $\alpha \approx$ 2.002118778953827

Methods	$\lvert x_1 - \alpha \rvert$	$\lvert x_2 - \alpha \rvert$	$\lvert x_3 - \alpha \rvert$
Newton Method	0.85925e-01	0.32675e-02	0.50032e-05
Weerakoon, p = 1 in (3)	0.18271e-01	0.14770e-05	0.0
Homeier, p = -1 in (3)	0.49772e-02	0.33027e-08	0.0
Lukic, p = 0 in (3)	0.11670e-01	0.21544e-06	0.0
Khattri (5)	0.14964e-01	0.45484e-07	0.0
Soleymani (17) in [9]	0.26594e-01	0.32982e-06	0.0
Jaiswal (2.25) in [3]	0.76770e-02	0.12105e-08	0.0
Method I - (14)	0.27318e-01	0.39052e-06	0.0
Method II - (16)	0.23844e-01	0.13910e-06	0.0
Method III - (20)	0.83302e-02	0.18560e-08	0.0
Method IV - (22)	0.25895e-01	0.27772e-06	0.0

Note: NM 6 itr, and other 4.

Table 3: The function $f_3(x) = \cos x + \sin 2x\sqrt{1 - x^2} + \sin x^2 + x^{14} + x^3 + \dfrac{1}{2x}$ by the methods described occurring errors with initial guess $x_0 = -0.9$ and the root is $\alpha \approx$ - 0.9257722498275615

Methods	$\lvert x_1 - \alpha \rvert$	$\lvert x_2 - \alpha \rvert$	$\lvert x_3 - \alpha \rvert$
Newton Method	0.46824e-02	0.14113e-03	0.12997e-06
Weerakoon, p = 1 in (3)	0.11709e-02	0.95452e-07	0.0
Homeier, p = -1 in (3)	0.22725e-03	0.19248e-09	0.0
Lukic, p = 0 in (3)	0.70352e-03	0.13169e-07	0.11102e-15
Khattri (5)	0.23653e-02	0.71183e-07	0.0
Soleymani (17) in [9]	0.55756e-03	0.78787e-10	0.11102e-15
Jaiswal (2.25) in [3]	0.55682e-03	0.78274e-10	0.0
Method I - (14)	0.48788e-03	0.40454e-10	0.0

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Method II - (16)	0.11602e-02	0.24333e-08	0.0
Method III - (20)	0.62711e-03	0.14153e-09	0.0
Method IV - (22)	0.62786e-03	0.14236e-09	0.0

Note: NM 6 itr, and other 4.

Table 4: The function $f_4(x) = \ln(x) - x^3 + 2\sin x$ by the methods described occurring errors with initial guess $x_0 = 1.5$. The root is $\alpha \approx 1.297997743280372$

Methods	$\lvert x_1 - \alpha \rvert$	$\lvert x_2 - \alpha \rvert$	$\lvert x_3 - \alpha \rvert$
Newton Method	0.37989e-01	0.18210e-02	0.45438e-05
Weerakoon, p = 1 in (3)	0.86903e-02	0.12841e-05	0.0
Homeier, p = -1 in (3)	0.23183e-02	0.16189e-08	0.0
Lukic, p = 0 in (3)	0.55301e-02	0.17888e-06	0.22204e-15
Khattri (5)	0.76827e-02	0.77927e-07	0.0
Soleymani (17) in [9]	0.32801e-02	0.92473e-09	0.22204e-15
Jaiswal (2.25) in [3]	0.41530e-02	0.26516e-08	2.22044e-15
Method I - (14)	0.29423e-02	0.52997e-09	0.0
Method II - (16)	0.45316e-02	0.54138e-08	0.0
Method III - (20)	0.44766e-02	0.39426e-08	0.22204e-15
Method IV - (22)	0.36051e-02	0.15034e-08	0.0

Note: NM 6 itr, and other 4.

Table 5: The function $f_5(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$ by the methods described occurring errors with initial guess $x_0 = -2$. The root is $\alpha \approx -1.207647827130919$

Methods	$\lvert x_3 - \alpha \rvert$	$\lvert x_4 - \alpha \rvert$	$\lvert x_5 - \alpha \rvert$
Newton Method	0.15932e-00	0.35638e-01	0.18836e-02
Weerakoon, p = 1 in (3)	0.13926e-01	0.86676e-05	0.19984e-14
Homeier, p = -1 in (3)	0.30882e-03	0.29496e-10	0.22204e-15
Lukic, p = 0 in (3)	0.32211e-02	0.71044e-07	0.22204e-15
Khattri (5)	0.13814e-01	0.98872e-06	0.22204e-15
Soleymani (17) in [9]	0.68539e-03	0.23501e-11	0.22204e-15
Jaiswal (2.25) in [3]	0.51094e-03	0.65814e-12	0.22204e-15
Method I - (14)	0.34251e-03	0.12967e-12	0.22204e-15
Method II - (16)	0.26764e-02	0.88511e-09	0.22204e-15
Method III - (20)	0.90973e-03	0.74649e-11	0.22204e-15
Method IV - (22)	0.11813e-02	0.23145e-10	0.22204e-15

Note: NM 9 itr, and other 6.

7. Conclusions

Based on power mean Newton's method, we have presented two class of modified Newton's method MNM1 and MNM2 in this paper. These new methods have sixth order convergence and have efficiency index $E^* = 1.431$ and 1.565 respectively. As per efficiency is concerned, the method MNM2 performs equally well with few existing sixth order methods [2, 7, 8] but better than Newton's method and power means Newton's method [15]. Also, by combining power means Newton method with the well-known Newton's method in a three-step iteration process, we can get few existing sixth order methods as special cases of the present method. Hence, the present methods can be considered as a class of sixth order Newton's methods.

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