

Chapter 5

Simplified Proofs for Properties of Square Roots of Real Numbers

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Abstract

The properties of square roots are of significant interest in Number Theory. This paper covers the general case of square roots of real numbers. Numerous properties of such square roots are proven in a straightforward manner and, for cases in which $\sqrt{x}$ is rational, elementary calculation methods are presented to determine the values of $\sqrt{x}$.

Keywords: Number theory, Prime factorization, Square roots, Important properties of roots, Elementary calculation methods for root values.

1. Introduction

In numerous publications during the recent past (e.g. [7], [9]), we intensively studied the question “For which values of $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $n\sqrt{x}$ is still a rational number?”. We were able to completely answer the question for all $x \in \mathbb{R}$, if x is not an extremely large number.

Evidently, our publications, in particular, cover the special case of square roots (for $n=2$). Typically, properties of square roots are of significantly more interest as compared to the more general roots in which $n \neq 2$. So, we decided to discuss the implications of our general results (covering the cases for arbitrary $n \in \mathbb{N}$) to the case of square roots $2\sqrt{x}$ (or $\sqrt{x}$ for short). Astonishingly, quite a few of our proofs for general $n\sqrt{x}$ become considerably simpler for square roots.

The paper is structured as follows: Section 2 analyses the properties of $\sqrt{x}$ for $x \in \mathbb{R}^-$. (Remark: To simplify our presentation in this paper let us introduce the notation $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x \geq 0\}$, $\mathbb{R}^- := \{x \in \mathbb{R} \mid x < 0\}$ with analogous definitions for $\mathbb{Q}^+$ and $\mathbb{Q}^-$). Section 3 covers the case $\sqrt{x}$ for $x \in \mathbb{R}^+ \setminus \mathbb{Q}$. In Section 4 we treat $\sqrt{x}$, assuming $x \in \mathbb{N}$. Section 5 finally presents results for square roots $\sqrt{x}$, if $x \in \mathbb{Q}^+ \setminus \mathbb{N}$. Section 6 will conclude this paper by giving a short outlook.

2. Properties of $\sqrt{x}$ for $x \in \mathbb{R}^-$

Let us assume that for a given $x \in \mathbb{R}^-$ there exists a $y = \sqrt{x}$. Then, $y = \sqrt{x}$ implies $y^2 = x < 0$. However, for all $y \in \mathbb{R}$ it is evident that $y^2 \geq 0$. Therefore, $\sqrt{x} \in \mathbb{C} \setminus \mathbb{R}$ if $x \in \mathbb{R}^-$. This implies, in particular, that $\sqrt{x} \notin \mathbb{Q} \forall x \in \mathbb{R}^-$.

3. Properties of $\sqrt{x}$ for $x \in \mathbb{R}^+ \setminus \mathbb{Q}$

In [7] it has been proven that $\forall x \in \mathbb{R}^+ \setminus \mathbb{Q}$ and $\forall n \in \mathbb{N} : n\sqrt{x} \notin \mathbb{Q}$ holds. Therefore, if $x \in \mathbb{R}^+ \setminus \mathbb{Q}$ is assumed, we can conclude that $\sqrt{x} \notin \mathbb{Q} \forall x \in \mathbb{R}^+ \setminus \mathbb{Q}$ because we just consider the special case $n=2$.

4. Properties of $\sqrt{x}$ for $x \in \mathbb{N}$

As suggested in [7] we use the (unique) prime factorization of x , [1], [2], $x \in \mathbb{N}$ to directly answer the question “Does $\sqrt{x} \in \mathbb{Q}$ or $\sqrt{x} \notin \mathbb{Q}$ hold?”

Very pleasingly, at present, already valuable software is available in the Internet providing prime factorization for a large number of integers (see, e.g., the highly efficient app developed by W. Fendt [3] which provides factorization for numerous values of x , namely $1 \leq x \leq 10^{12}$). Astonishingly, in most cases, those factorization results are delivered within just a few seconds.

Let the prime factorization of x be:

$$x = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}, \quad (1)$$

p_i denoting (mutually different) primes $\forall i \in \{1, \dots, m\}$, $m, k_i \in \mathbb{N}$.

Remark:

Evidently, if $x=1$ then the prime factorization of x is undefined because the number “1” is neither a prime number nor a product of primes. Anyway, here we will introduce the definition: “The (pseudo) prime number factorization of 1 is defined to be $1=2^0$ ”. As the exponent of 2 is 0 in this definition, this exponent is an even number and therefore we do not have to exclude the value $x=1$ when we study $\sqrt{x}$ in Sections 4 and 5, i.e. we just apply our simplifying definition for prime factorization of “1” in case of $x=1$ (in Section 4) and if $r, s=1$ (in Section 5).

In [3] it has been demonstrated that, for given $n \in \mathbb{N}$:

$$n\sqrt{x} \in \mathbb{Q} \iff k_i \text{ is an integer multiple of } n \forall i \in \{1, 2, \dots, m\}. \quad (2)$$

As, here, we are interested in the rationality of square roots we need to apply Equation (2) for the special case $n=2$. This leads to the following condition for square roots:

Theorem:

For given $x \in \mathbb{N}$ with prime factorization according to Equation (1) the following equivalency condition holds:

$$\sqrt{x} \in \mathbb{Q} \iff k_i \text{ is an even number } \forall i \in \{1, 2, \dots, m\}. \quad (3)$$

Proof:

As Equation (2) has been proven for arbitrary $n \in \mathbb{N}$, $n=2$ is just a trivial special case. $\square$

Also, the task to determine the value of $\sqrt{x}$ if $\sqrt{x} \in \mathbb{Q}$ is straightforward indeed (for details, see [10]):

If $\sqrt{x} \in \mathbb{Q}$, with $x \in \mathbb{N}$ the value(s) of $\sqrt{x}$ are just given by:

$$\sqrt{x} = \pm p_1^{\frac{k_1}{2}} \cdot p_2^{\frac{k_2}{2}} \cdots p_i^{\frac{k_i}{2}} \cdots p_m^{\frac{k_m}{2}}. \quad (4)$$

Evidently, if $\sqrt{x} \in \mathbb{Q}$, with $x \in \mathbb{N}$, then Equation (4) implies: $\sqrt{x} \in \mathbb{Z}$

A direct consequence of this Theorem is the following simple necessary and sufficient condition:

$$\sqrt{x} \notin \mathbb{Q} \iff \exists k_j (\text{at least one!}) \text{ in the prime factorization of } x \text{ such that } k_j \text{ is an odd number.} \quad (5)$$

In particular, Equation (5) implies that we do not need to have complete knowledge regarding the prime factorization of x to conclude that $\sqrt{x} \notin \mathbb{Q}$. It is sufficient already to know that this prime factorization contains at least one prime factor whose exponent is odd. Regarding the general case $n\sqrt{x}$, where $n \in \mathbb{N}$, please see e.g. [11]. Moreover, two extremely simple tests can be applied to prove that $\sqrt{x} \notin \mathbb{Q}$, for details see [8]:

- **Test 1:** The last two digits of x are no integer multiple of 4 (i.e. of 2^2) AND x is an even number, i.e. x is an integer multiple of 2.
- **Test 2:** The last two digits of x are unequal $\dots 00, \dots 25, \dots 50, \dots 75$ and therefore x is no integer multiple of 25 (i.e. of 5^2) AND the last digit of x is 0 or 5, i.e. x is an integer multiple of 5.

5. Properties of $\sqrt{x}$ for $x \in \mathbb{Q}^+ \setminus \mathbb{N}$

As, here, x is assumed to satisfy $x \in \mathbb{Q}^+ \setminus \mathbb{N}$ we can conclude that $\exists r, s \in \mathbb{N}$ such that $x = \frac{r}{s}$. Let, moreover, r and s be co-prime and let

$$r = r_1^{\alpha_1} \cdot r_2^{\alpha_2} \cdots r_v^{\alpha_v} \text{ and } s = s_1^{\beta_1} \cdot s_2^{\beta_2} \cdots s_\mu^{\beta_\mu} \quad (6)$$

be the prime factorizations of r and s , where $r_i, i \in \{1, 2, \dots, v\}$ and $s_j, j \in \{1, 2, \dots, \mu\}$ are prime numbers, $v, \mu \in \mathbb{N}$.

Here, again $\sqrt{x}$ represents a special case of $n\sqrt{x}$ (for $n=2$). And, therefore, the corresponding theorem derived in [7] simplifies to the following

Theorem:

For given $x = \frac{r}{s} \in Q^+ \setminus N$ with prime factorizations according to Equation (6) the following equivalency condition holds:

$$\sqrt{x} = \sqrt{\frac{r}{s}} \in Q \iff \alpha_i \text{ and } \beta_j \text{ are even numbers } \forall i \in \{1, 2, \dots, v\} \text{ and } \forall j \in \{1, 2, \dots, \mu\} \quad (7)$$

Proof:

As $n=2$, the theorem represents a special case of the more general case covered by [7]. $\square$

Again, a necessary and sufficient condition for $\sqrt{x} = \sqrt{\frac{r}{s}} \notin Q$ here is that there exists (at least) one α_i or β_j which is an odd number. Moreover, if $\sqrt{x} = \sqrt{\frac{r}{s}} \in Q$ the values of $\sqrt{x}$ can be obtained directly and we get

$$\sqrt{x} = \pm \frac{r_1^{\frac{\alpha_1}{2}} \cdot r_2^{\frac{\alpha_2}{2}} \cdots r_v^{\frac{\alpha_v}{2}}}{s_1^{\frac{\beta_1}{2}} \cdot s_2^{\frac{\beta_2}{2}} \cdots s_\mu^{\frac{\beta_\mu}{2}}} \quad (8)$$

For details regarding the more general case $n\sqrt{x}$, where $n \in N$, the reader is referred to [10].

6. Conclusions and Outlook

The results presented in this paper allow us, for any $x \in R$, to decide definitively whether the value of $\sqrt{x}$ is rational or irrational provided that prime factorization of x according to Equation (1) or Equation (6) can be determined. Alternatively, to just prove " $\sqrt{x} \notin Q$ ", it is sufficient to find a prime factor in the factorization of x the exponent of which is an odd number (if $x \in N$) or – if $x \in Q^+ \setminus N$ – to find (at least) one prime number with an odd exponent in the factorization of the numerator or the denominator of x . Therefore, we are able to answer the question " $\sqrt{x} \in Q$ or $\sqrt{x} \notin Q$?" for a huge number of values of x . One of the results obtained by us is, that a necessary condition for $\sqrt{x} \in Q$ is: $x \in Q^+$.

Anyway, values of x can be constructed for which it is impossible to determine whether $\sqrt{x} \in Q$ – if we just make use of the methods presented here by us. An example for such an (extremely large) value of x for which the decision " $\sqrt{x} \in Q$ or $\sqrt{x} \notin Q$?" is still impossible currently (from our point of view) is, e.g., $x = p_1 \cdot p_2$, where p_1 and p_2 represent two different, extremely large prime numbers.

Such values of x could be too large, by far, to obtain the corresponding prime factorization of x (see, e.g., [4], [5]). Besides it is interesting to note that the practical impossibility to obtain the prime factors of a number which represents the product of two extremely large prime numbers has led to the invention of an important encryption algorithm which is often used in the domain of (computer) network security, namely, the RSA algorithm [6].

It will be strongly interesting to see, whether future research will indeed be able to even cover the square roots of such extremely large values of x .

Declaration

Competing Interests: Author has declared that no competing interests exist.

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